Entanglement of Bell Mixtures of Two Qubits

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Abstract

This paper is an appendix to a previous paper: quant-ph/0101123 "Relaxation Method for Calculating Quantum Entanglement", by Robert Tucci. For certain mixtures of Bell basis states, namely the Werner States, we use the theoretical machinery of our previous paper to derive algebraic formulas for: the pure and mixed minimization entanglements (i.e., E_{pure} and E_{mixed}), their optimal decompositions and their entanglement operators. This complements and corroborates some results that were obtained numerically but not algebraically in our previous paper. Some of the algebraic formulas presented here are new. Others were first derived using a different method by Bennett et al in quant-ph/9604024.

1 Introduction

This paper is an appendix to a previous paper[1] by the same author. We will assume that the reader has read our previous paper. Without having done so, he/she won't be able to understand this paper beyond its Introduction.

Henceforth, we will use "min." as an abbreviation for the word "minimization". In our previous paper[1], we defined two quantum entanglement measures, the *pure min. entanglement* (E_{pure}) and the *mixed min. entanglement* (E_{mixed}). These measures apply to any bipartite density matrix (the subscripts refer to the type of minimization space used, not to whether the density matrix is pure or mixed.) We showed that E_{pure} is equal to the entanglement of formation, a measure of entanglement first defined by Bennett et al in Ref.[2]. E_{mixed}, on the other hand, is a new animal. We gave a numerical method for calculating E_{pure} and E_{mixed}, their optimal decompositions, and also their entanglement operators (operators whose expectation value gives the entanglement). We gave numerical results obtained with Causa Común, a computer program that implements the ideas of Ref.[1]. We did this for a special type of Bell mixture called a Werner State and for Horodecki States that exhibit bound entanglement.

In Ref.[2], Bennett et al derived an explicit algebraic formula for the entanglement of formation of any Bell mixture. In Ref.[3], Wootters went one step further and generalized the formula of Ref.[2] to encompass all density matrices of two qubits.

In this paper, we use the theoretical machinery of our previous paper to derive certain algebraic formulas for Werner States. Specifically, we give explicit algebraic formulas for E_{pure} and E_{mixed}, their optimal decompositions and their entanglement operators. This complements and corroborates some results that were obtained numerically but not algebraically in our previous paper. Most of our formulas for E_{pure} were first derived, using a different method, by Bennett et al in Ref.[2]. Our formulas for E_{mixed} are new.

2 Notation

We assume the reader is familiar with the notation of Ref.[1]. In this section we will introduce some additional notation that is used throughout this paper.

We will use the notation of Ref.[1] intact except for one small modification. Ref.[1] dealt with a Hilbert space $\mathcal{H}_{\underline{xy}} = \mathcal{H}_{\underline{x}} \otimes \mathcal{H}_{\underline{y}}$. Its two parts were represented by the random variables $\underline{x}$ and $\underline{y}$ (Xerxes and Yolanda). Here we will rename the two parts $\underline{a}$, $\underline{b}$ (Alice and Bob). This conforms more closely with the rest of the literature. Also, it looks better in cases such as the one considered in this paper where one also uses x, y, z for indices of Pauli matrices. In conclusion, throughout this paper, we will be dealing with $\mathcal{H}_{\underline{ab}}$ where $S_{\underline{a}} = S_{\underline{b}} = Bool$.

For any Hilbert space $\mathcal{H}$ and any $|\psi\rangle \in \mathcal{H}$, we will often represent the projection operator $|\psi\rangle\langle\psi|$ by $\pi(\psi)$. $\mathcal{L}(\mathcal{H})$ will denote the set of linear operators acting on

$\mathcal{H}$.

Let $Z_{j,k}$ be the set of integers from j to k, including both j and k. Let $Bool = \{0, 1\}$. Let $x^{\#n}$ be the n-tuple with x repeated n times. For example, $0^{\#3} = 0, 0, 0$.

The Kronecker delta function $\delta(x, y)$ equals one if $x = y$ and zero otherwise. We will often abbreviate $\delta(x, y)$ by δ^x_y, $\delta(x, y)\delta(p, q)$ by δ^{xp}_{yq}, etc. Also, we will use $\delta^{\widehat{\nu}}_{\mu}$ as an abbreviation for $1 - \delta^{\nu}_{\mu}$. In other words, $\delta^{\widehat{\nu}}_{\mu}$ is an an indicator function which equals 1 whenever $\mu \neq \nu$ and zero when $\mu = \nu$. For example, if $\mu, \nu \in Z_{0,3}$, then the metric in Special Relativity can be written as $g_{\mu\nu} = (\delta^{00}_{\mu\nu} - \delta^{\widehat{00}}_{\mu\nu})\delta^{\mu}_{\nu}$. One must be careful not to confuse $\delta^{\widehat{\alpha}\widehat{\beta}}_{\mu\nu} = (1 - \delta^{\alpha}_{\mu})(1 - \delta^{\beta}_{\nu})$ and $\delta^{\widehat{\alpha\beta}}_{\mu\nu} = 1 - \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}$.

We will often use the *color summation convention*[4]. By this we mean that the summation signs will not be shown; summation will instead be indicated by displaying summed indices in a different color than the unsummed ones. For example, $F_{\mu\nu}v^{\nu} = \sum_{\nu} F_{\mu\nu}v^{\nu}$. This is a better notation than the *Einstein implicit summation convention* which it is meant to replace. In the Einstein convention, we are instructed to sum over repeated indices. This becomes clumsy and requires a warning to the reader whenever we wish to use repeated indices that are not summed over.

As is common in Relativity texts, we will often use Greek letters to represent indices that range over $Z_{0,3}$ and Latin letters to represent indices that range over $Z_{1,3}$. Unlike Relativity texts, we will not distinguish between upper and lower indices.

Define $\mathcal{F}_2 = diag(1, -1, 1)$. For any 3-dimensional vector $\vec{n} = (n_1, n_2, n_3)^T$, $\mathcal{F}_2\vec{n} = (n_1, -n_2, n_3)^T$. One can likewise define $\mathcal{F}_j$ for $j \in Z_{1,3}$ to be an operator that "flips" the jth component of the vector it acts on.

For any 3-dimensional vectors $\vec{a}$ and $\vec{x}$, $\vec{a} \times \vec{x} = A\vec{x}$, where

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} . \tag{1}$$

We will often represent A by $[\vec{a} \times \cdot]$.

Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2}$$

Let $\sigma^0 = 1$. Let $\vec{\sigma}$ be the 3 dimensional vector of Pauli matrices. The Pauli matrices are defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{3}$$

As is well known, the Pauli matrices satisfy:

$$\sigma^k\sigma^r = \delta^r_k + i\epsilon_{krj}\sigma^j , \tag{4}$$

for $k, r, j \in Z_{1,3}$, where ϵ_{krj} is the totally antisymmetric tensor with $\epsilon_{123} = 1$. Unfortunately, there is no formula for $\sigma^\mu \sigma^\nu$ with $\mu, \nu \in Z_{0,3}$, that matches the conciseness and standardization of Eq.(4). Here is one particular attempt.

$$\sigma^\mu \sigma^\nu = f_{\mu\nu} \sigma^{\mu \oplus \nu} \ , \tag{5}$$

where

$$\mu \oplus \nu = \begin{array}{c|cccc|c} & 0 & 1 & 2 & 3 & \nu \\ \hline 0 & 0 & 1 & 2 & 3 & \\ 1 & 1 & 0 & 3 & 2 & \\ 2 & 2 & 3 & 0 & 1 & \\ 3 & 3 & 2 & 1 & 0 & \\ \mu & & & & & \end{array} \ , \tag{6}$$

and

$$f_{\mu\nu} = \begin{array}{c|cccc|c} & 0 & 1 & 2 & 3 & \nu \\ \hline 0 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & i & -i & \\ 2 & 1 & -i & 1 & i & \\ 3 & 1 & i & -i & 1 & \\ \mu & & & & & \end{array} \ . \tag{7}$$

Note that the operation $\mu \oplus \nu$ defined by Eq.(6) specifies an Abelian group (the operation is commutative, associative, has an identity, and has an inverse for each of its elements). The Abelian group defined by $\oplus$ on $Z_{0,3}$ can be shown to be simply the product of two copies of the group of two elements.

Instead of defining $f_{\mu\nu}$ by the table Eq.(7), one can define it by the rather clumsy expression:

$$f_{\mu\nu} = \delta^\nu_\mu + \delta^{\widehat{00}}_{\mu\nu} + \delta^{0\widehat{0}}_{\mu\nu} + \delta^{\widehat{0\,0}}_{\mu\nu} i \epsilon_{\mu,\nu,\mu\oplus\nu} \ . \tag{8}$$

$f_{\mu\nu}$ has a few useful properties. For example, it is Hermitian and it satisfies:

$$f^*_{\alpha\oplus\beta,\beta} = f_{\alpha,\beta} \ . \tag{9}$$

For any $x \in [0, 1]$, the *binary entropy function* $h(x)$ is defined by

$$h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x) \ . \tag{10}$$

Occasionally, we will also need to use $h(x)$ with the base 2 logs replaced by base e ones. So define

$$h_e(x) = (\ln 2) h(x) = -x \ln(x) - (1 - x) \ln(1 - x) \ . \tag{11}$$

3 Bell Basis

In this section we will discuss various properties of the Bell Basis.

One can define operators that act only on the $\mathcal{H}_{\underline{a}}$ (ditto, $\mathcal{H}_{\underline{b}}$) part of $\mathcal{H}_{\underline{ab}}$. Let A^μ (ditto, B^μ) for $\mu \in Z_{0,3}$ represent the Pauli matrices that act on space $\mathcal{H}_{\underline{a}}$ (ditto, $\mathcal{H}_{\underline{b}}$). Another natural notation for these operators is $\sigma^\mu_{\underline{a}}$ and $\sigma^\mu_{\underline{b}}$.

The following four states are usually called the "Bell basis" (with the magic phases) of $\mathcal{H}_{\underline{ab}}$:

$$|B(0)\rangle = |=^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \ , \tag{12}$$

$$|B(1)\rangle = iB^1|=^+\rangle = \frac{i}{\sqrt{2}}(|01\rangle + |10\rangle) = i|\neq^+\rangle \ , \tag{13}$$

$$|B(2)\rangle = iB^2|=^+\rangle = \frac{-1}{\sqrt{2}}(|01\rangle - |10\rangle) = -|\neq^-\rangle \ , \tag{14}$$

$$|B(3)\rangle = iB^3|=^+\rangle = \frac{i}{\sqrt{2}}(|00\rangle - |11\rangle) = i|=^-\rangle \ . \tag{15}$$

(By taking matrix products and linear combinations with **real** coefficients, of the operators 1, $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$, one generates what is called the Quaternion Algebra, invented by Hamilton.)

The Bell basis states are an orthonormal basis of $\mathcal{H}_{\underline{ab}}$ so they satisfy

$$\langle B(\mu)|B(\nu)\rangle = \delta^\mu_\nu \ , \tag{16}$$

and

$$|B(\mu)\rangle\langle B(\mu)| = 1 \ . \tag{17}$$

The Bell basis states place listeners $\underline{a}$ and $\underline{b}$ on equal footing: measurement of A^μ is the same as measurement of B^μ up to a sign. Indeed, the action of A^μ on $|B(0)\rangle$ is

$$A^\mu|=^+\rangle = (-1)^{\delta^2_\mu}B^\mu|=^+\rangle \ . \tag{18}$$

The action of A^μ on $|B(\nu)\rangle$ for $\nu \neq 0$ may have an additional -1 factor due to the fact that the Pauli matrices anticommute. For example,

$$\begin{aligned} A^2|B(3)\rangle &= A^2iB^3|=^+\rangle = iB^3A^2|=^+\rangle = -iB^3B^2|=^+\rangle = \\ &= B^2iB^3|=^+\rangle = B^2|B(3)\rangle \ . \end{aligned} \tag{19}$$

Thus, we see that, in general, the action of A^μ on $|B(\nu)\rangle$ is

$$A^\mu|B(\nu)\rangle = (-1)^{\delta^2_\mu}(-1)^{\widehat{\delta^{00}_{\mu\nu}}\widehat{\delta^\mu_\nu}}B^\mu|B(\nu)\rangle \ . \tag{20}$$

Suppose $\Omega_{\underline{b}}$ (ditto, $\Omega_{\underline{a}}$) is an operator acting only on $\mathcal{H}_{\underline{b}}$ (ditto, $\mathcal{H}_{\underline{a}}$). We will call such operators *local* operators. It is easy to show that

$$\langle B(\mu)|\Omega_{\underline{b}}|B(\mu)\rangle = \langle =^+ |\Omega_{\underline{b}}| =^+\rangle = \frac{1}{2}\mathrm{tr}(\Omega_{\underline{b}}) \ , \tag{21}$$

and

$$\langle B(\mu)|\Omega_{\underline{a}}|B(\mu)\rangle = \langle =^+ |\Omega_{\underline{a}}| =^+\rangle = \frac{1}{2}\mathrm{tr}(\Omega_{\underline{a}}) \ . \tag{22}$$

Note the right hand sides are independent of μ (although $\langle B(\mu)|\Omega_{\underline{b}}|B(\nu)\rangle$ generally does depend on μ and ν.) Hence, all 4 Bell basis states contain the same amount of information about expected values of local operators.

In future sections, we will need to find the matrix elements in the Bell basis of certain operators in $\mathcal{L}(\mathcal{H}_{\underline{ab}})$. These operators can always be written as a linear combination $x_{\mu\nu}A^\mu B^\nu$. In this linear combination, A^μ will be acting on a Bell state so it can be replaced by plus or minus B^μ. The product $B^\mu B^\nu$ can itself be replaced by $f_{\mu\nu}B^{\mu\oplus\nu}$. In this way, we can reduced the problem of calculating the matrix elements in the Bell basis of any operator in $\mathcal{L}(\mathcal{H}_{\underline{ab}})$ to calculating the matrix elements in the Bell basis of B^β. One has:

$$\begin{aligned}\langle B(\mu_1)|B^\beta|B(\mu_2)\rangle &= (-i)^{\widehat{\delta^0_{\mu_1}}} f_{\mu_1,\beta\oplus\mu_2} f_{\beta,\mu_2} (i)^{\widehat{\delta^0_{\mu_2}}} \langle =^+ |B^{\beta\oplus\mu_1\oplus\mu 2}| =^+\rangle \\ &= (-i)^{\widehat{\delta^0_{\mu_1}}} f_{\beta,\mu_2} (i)^{\widehat{\delta^0_{\mu_2}}} \delta^0_{\beta\oplus\mu_1\oplus\mu 2} \ . \end{aligned} \tag{23}$$

An equivalent way of writing the last equation is:

$$\langle B(\mu_1)|B^0|B(\mu_2)\rangle = \delta^{\mu_1}_{\mu_2} \ , \tag{24}$$

and

$$\langle B(\mu_1)|\vec{x}\cdot\vec{B}|B(\mu_2)\rangle = \left[\begin{array}{cc} 0 & i\vec{x}^T \\ -i\vec{x} & i(\vec{x}\times\cdot) \end{array}\right]_{\mu_1,\mu_2} \ . \tag{25}$$

Another problem that we shall encounter in future sections is finding the partial trace with respect to either $\underline{a}$ or $\underline{b}$ of an operator $X \in \mathcal{L}(\mathcal{H}_{\underline{ab}})$. If

$$X = x_{\mu\nu}|B(\mu)\rangle\langle B(\nu)| \ , \tag{26}$$

then one finds that

$$\mathrm{tr}_{\underline{a}}X = \frac{1}{2}\left\{x_{\mu\mu} + [x_{k0} - x_{0k} + x_{pq}\epsilon_{pqk}]\, iB^k\right\} \ , \tag{27}$$

and

$$\mathrm{tr}_{\underline{b}}X = \frac{1}{2}\left\{y_{\mu\mu} + [y_{k0} - y_{0k} + y_{pq}\epsilon_{pqk}]\, iA^k\right\} \ , \tag{28}$$

where $y_{\mu\nu} = x_{\mu\nu}(-1)^{\delta^2_\mu}(-1)^{\delta^2_\nu}$. These equations generalize the well known result:

$$\mathrm{tr}_{\underline{a}}|B(\mu)\rangle\langle B(\mu)| = \frac{1}{2}\left(\pi(|0\rangle_{\underline{b}}) + \pi(|1\rangle_{\underline{b}})\right) \ , \tag{29a}$$

$$\mathrm{tr}_{\underline{b}}|B(\mu)\rangle\langle B(\mu)| = \frac{1}{2}\left(\pi(|0\rangle_{\underline{a}}) + \pi(|1\rangle_{\underline{a}})\right) \ . \tag{29b}$$

Thus, the Bell basis states do not distinguish between $|0\rangle_{\underline{a}}$ and $|1\rangle_{\underline{a}}$ (ditto, $|0\rangle_{\underline{b}}$ and $|1\rangle_{\underline{b}}$) when the state of $\underline{b}$ (ditto, $\underline{a}$) is unknown. An immediate consequence of the last equation is that the Bell states have maximum entanglement of formation.

An element of $\mathcal{L}(\mathcal{H}_{\underline{ab}})$ can be expanded in various bases: one can expand it in terms of the operators $A^\mu B^\nu$ for all $\mu, \nu \in Z_{0,3}$ (call this the Pauli $\mathcal{L}(\mathcal{H}_{\underline{ab}})$-basis), or the operators $|a, b\rangle\langle a', b'|$ for all $a, b, a', b' \in Bool$ (call this the Standard $\mathcal{L}(\mathcal{H}_{\underline{ab}})$-basis), or the operators $|B(\mu)\rangle\langle B(\nu)|$ for all $\mu, \nu \in Z_{0,3}$ (call this the Bell $\mathcal{L}(\mathcal{H}_{\underline{ab}})$-basis). In what follows, we will use mostly the Bell $\mathcal{L}(\mathcal{H}_{\underline{ab}})$-basis because it seems the most natural one for doing entanglement calculations. Thus, henceforth, whenever we represent $\mathcal{L}(\mathcal{H}_{\underline{ab}})$ operators by 4 by 4 matrices, the matrices should be understood as representations in the Bell $\mathcal{L}(\mathcal{H}_{\underline{ab}})$-basis.

4 Entanglement of Pure State

In this section we calculate the entanglement of any pure state of two qubits[2]. This is a good warm up exercise to prepare us for the following sections, where we address the harder problem of calculating entanglements of mixed states.

Below, for any complex vector $\vec{z}$, we will use $|\vec{z}| = \sqrt{\vec{z}\cdot\vec{z}^*}$ and $\vec{z}^{\,2} = \vec{z}\cdot\vec{z}$.

Any unit length $|\psi\rangle \in \mathcal{H}_{\underline{ab}}$ can be expressed in the Bell basis as:

$$|\psi\rangle = (z_0 + i\vec{z}\cdot\vec{B})|=^+\rangle \ , \tag{30}$$

where $\langle\psi|\psi\rangle = |z_0|^2 + |\vec{z}|^2 = 1$. If

$$\rho = |\psi\rangle\langle\psi| = (z_0 + i\vec{z}\cdot\vec{B})|=^+\rangle\langle=^+|(z_0^* - i\vec{z}^*\cdot\vec{B}) \ , \tag{31}$$

then

$$\mathrm{tr}_{\underline{a}}\rho = \frac{1}{2}(z_0 + i\vec{z}\cdot\vec{B})(z_0^* - i\vec{z}^*\cdot\vec{B}) = n_0 + \vec{n}\cdot\vec{B} \ , \tag{32}$$

where

$$n_0 = \frac{1}{2} \ , \tag{33}$$

and

$$\vec{n} = \frac{i}{2}(z_0^*\vec{z} - z_0\vec{z}^* + \vec{z}\times\vec{z}^*) \ . \tag{34}$$

From Eq.(32) and Appendix A, the eigenvalues of $\mathrm{tr}_{\underline{a}}\rho$ are simply $n_0 \pm |\vec{n}|$. Hence,

$$E_{pure} = E_{mixed} = h(n_0 + |\vec{n}|) \ . \tag{35}$$

One can show using well known vector product identities that for any 4-tuple $(z_0, \vec{z})$ of complex numbers such that $|z_0|^2 + |\vec{z}|^2 = 1$, one has

$$|z_0^* \vec{z} - z_0 \vec{z}^{\,*} + \vec{z} \times \vec{z}^{\,*}|^2 = 1 - |z_0^2 + \vec{z}^{\,2}|^2 \ . \tag{36}$$

Hence

$$E_{pure} = E_{mixed} = h(\frac{1 + \sqrt{1 - C^2}}{2}) \ , \tag{37}$$

where

$$C = |z_0^2 + \vec{z}^{\,2}| \ . \tag{38}$$

C is called the *concurrence*[2] of $|\psi\rangle$. E_{pure} is a monotonically nondecreasing function of C, and they both vanish at the same time, so C is also a good measure of entanglement. $0 \leq C \leq 1$. The pure state $|\psi\rangle$ has $C = 1$ (maximum entanglement) iff its coefficients $(z_0, \vec{z})$ are all real.

5 Entanglement of Bell Mixture

In this section we present the main calculation of this paper. For Werner states, we calculate E_{pure} and E_{mixed}, and their corresponding optimal decompositions and entanglement operators. Our calculation is split into 4 parts: (1)$K^{\alpha}_{\underline{ab}}$, (2)$R^{\alpha}_{\underline{ab}}$ (3)E_{pure} and E_{mixed} (4)$\Delta_{\underline{ab}}$.

We will call a *Bell mixture* any density matrix $\rho_{\underline{ab}}$ that can be expressed as

$$\rho_{\underline{ab}} = \sum_{\mu} m_{\mu} |B(\mu)\rangle\langle B(\mu)| \ , \tag{39}$$

where $\sum_{\mu} m_{\mu} = 1$. We will call a *Werner state* any state that can be expressed as

$$\rho_{\underline{ab}} = m_0 |B(0)\rangle\langle B(0)| + m_1 \sum_{\mu=1}^{D_v} |B(\mu)\rangle\langle B(\mu)| \ , \tag{40}$$

where $D_v \in \{1, 2, 3\}$ and $m_0 + D_v m_1 = 1$. This is a slight generalization from what is commonly called a Werner state. The term Werner state usually refers to the case where $D_v = 3$ and the rank of $\rho_{\underline{ab}}$ is 4.

5.1 $K^{\alpha}_{\underline{ab}}$ Calculations

We will begin by assuming that $K^{\alpha}_{\underline{ab}}$ can be expressed in a special "ansatz" form.

Define

$$I_v = diag(1^{\#D_v}, 0^{\#3-D_v}) \ , \tag{41}$$

where $D_v \in Z_{1,3}$ was defined previously. Now assume $K^{\alpha}_{\underline{ab}}$ can be expressed in the Bell representation as:

$$K^{\alpha}_{\underline{ab}} = \frac{1}{N_{\underline{\alpha}}} \left[\begin{array}{ll} m_0 & iq\vec{v}^{\alpha T} \\ -iq\vec{v}^{\alpha} & m_1\vec{v}^{\alpha}\vec{v}^{\alpha T} + \epsilon m_1(I_v - \vec{v}^{\alpha}\vec{v}^{\alpha T}) \end{array} \right] \ . \tag{42}$$

We assume that q and ϵ are real, $m_0 + D_v m_1 = 1$, and $\{\vec{v}^{\alpha} | \alpha \in Z_{1,N_{\underline{\alpha}}}\}$ is a set of real 3-dimensional vectors satisfying

$$\vec{v}^{\alpha T}\vec{v}^{\alpha} = D_v \tag{43a}$$

for all $\alpha \in Z_{1,N_{\underline{\alpha}}}$,

$$\sum_{\alpha=1}^{N_{\underline{\alpha}}} \vec{v}^{\alpha} = 0 \ , \tag{43b}$$

and

$$\sum_{\alpha=1}^{N_{\underline{\alpha}}} \vec{v}^{\alpha}\vec{v}^{\alpha T} = N_{\underline{\alpha}} I_v \ . \tag{43c}$$

Here are some examples of sets of $\vec{v}^{\alpha}$'s that satisfy Eqs.(43).

$$D_v = 1, \ N_{\underline{\alpha}} = 2, \quad \vec{v}^{\alpha} \in \left\{ \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right), \left(\begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right) \right\} \ , \tag{44a}$$

$$D_v = 2, \ N_{\underline{\alpha}} = 4, \quad \vec{v}^{\alpha} \in \left\{ \left(\begin{array}{c} (-1)^a \\ (-1)^b \\ 0 \end{array} \right) | a, b \in Bool \right\} \ , \tag{44b}$$

$$D_v = 3, \ N_{\underline{\alpha}} = 4, \quad \vec{v}^{\alpha} \in \left\{ \left(\begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \left(\begin{array}{c} 1 \\ -1 \\ -1 \end{array} \right), \left(\begin{array}{c} -1 \\ 1 \\ -1 \end{array} \right), \left(\begin{array}{c} -1 \\ -1 \\ 1 \end{array} \right) \right\} \ , \tag{44c}$$

$$D_v = 3, \ N_{\underline{\alpha}} = 8, \quad \vec{v}^{\alpha} \in \left\{ \left(\begin{array}{c} (-1)^a \\ (-1)^b \\ (-1)^c \end{array} \right) | a, b, c \in Bool \right\} \ . \tag{44d}$$

Note that

$$\rho_{\underline{ab}} = \sum_{\alpha} K^{\alpha}_{\underline{ab}} = diag(m_0, (m_1)^{\#D_v}, 0^{\#3-D_v}) \ , \tag{45}$$

and

$$w_{\alpha} = \mathrm{tr}_{\underline{ab}} K^{\alpha}_{\underline{ab}} = 1/N_{\underline{\alpha}} \ . \tag{46}$$

For E_{pure}, it is clear that we want $q = \sqrt{m_0 m_1}$ and $\epsilon = 0$ in Eq.(42). For E_{mixed}, we intend to find those values of q and ϵ that give the smallest possible conditional mutual information.

Our ansatz $K^{\alpha}_{\underline{ab}}$ given by Eq.(42) depends on the following parameters: m_0, m_1, D_v, $N_{\underline{\alpha}}$, q, ϵ and the $\vec{v}^{\alpha}$. Out of these primitive parameters, one can construct the following auxiliary parameters whose use will significantly shorten our subsequent formulas.

$$\eta = D_v - \epsilon(D_v - 1) \ , \tag{47a}$$

$$u = m_0 + \eta m_1 \ , \tag{47b}$$

$$k = \frac{m_0 - \eta m_1}{2} \ , \tag{47c}$$

$$Y = |q|\sqrt{D_v} \ , \tag{47d}$$

$$X = \sqrt{k^2 + Y^2} \ . \tag{47e}$$

Next we will find the eigenvalues of $K^{\alpha}_{\underline{ab}}$. This can be done by using the following well known formula. Suppose M is a square matrix that can be partitioned into 4 blocks A, B, R_1, R_2:

$$M = \begin{bmatrix} A & R_2 \\ R_1 & B \end{bmatrix} \ , \tag{48}$$

where the submatrices A and B are square but R_1 and R_2 need not be. Then one can show that

$$\det(M) = \det(A) \det(B - R_1 \frac{1}{A} R_2) \ . \tag{49}$$

One can use the last equation to find the eigenvalues of our ansatz $K^{\alpha}_{\underline{ab}}$. One finds (independent of α):

eigenvalue	*degeneracy*
$\lambda_+ = \left(\frac{u}{2} + X\right)/N_{\underline{\alpha}}$	1
$\lambda_- = \left(\frac{u}{2} - X\right)/N_{\underline{\alpha}}$	1
$\lambda_0 = \epsilon m_1/N_{\underline{\alpha}}$	$D_v - 1$
0	$3 - D_v$

. (50)

Note that since the eigenvalues of $K^{\alpha}_{\underline{ab}}$ must be non-negative, we must have $\epsilon \geq 0$ and $\frac{u}{2} - X \geq 0$. Since

$$(\frac{u}{2} - X)(\frac{u}{2} + X) = (\frac{u}{2})^2 - X^2 = m_0 m_1 \eta - q^2 D_v \ , \tag{51}$$

it follows that $|q| \leq \sqrt{m_0 m_1 \frac{\eta}{D_v}} \leq \sqrt{m_0 m_1}$. One can also assume without loss of generality that $q \geq 0$ since if $q < 0$, then one can replace $q \to -q$ and $\vec{v}^{\alpha} \to -\vec{v}^{\alpha}$ for all α.

We also need to know $\ln K^{\alpha}_{\underline{ab}}$. To calculate $\ln K^{\alpha}_{\underline{ab}}$, it is not enough to find the eigenvalues of $K^{\alpha}_{\underline{ab}}$; we also need to find its eigenvectors. Our technique for finding them is inspired by Appendix A, where we find the eigensystem of any 2 by 2 Hermitian matrix.

We begin by defining, for each α, three operators called E_{α}, Σ_{α} and $P^{(0)}_{\alpha}$:

$$E_{\alpha} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\vec{v}^{\alpha}\vec{v}^{\alpha T}}{D_v} \end{bmatrix} \ , \tag{52}$$

$$\Sigma_{\alpha} = \frac{1}{X} \begin{bmatrix} k & iq\vec{v}^{\alpha T} \\ -iq\vec{v}^{\alpha} & -k\frac{\vec{v}^{\alpha}\vec{v}^{\alpha T}}{D_v} \end{bmatrix} \ , \tag{53}$$

$$P^{(0)}_{\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & I_v - \frac{\vec{v}^{\alpha}\vec{v}^{\alpha T}}{D_v} \end{bmatrix} \ . \tag{54}$$

Note that these operators satisfy the following multiplication table:

	E_{α}	Σ_{α}	$P^{(0)}_{\alpha}$
E_{α}	E_{α}	Σ_{α}	0
Σ_{α}	Σ_{α}	E_{α}	0
$P^{(0)}_{\alpha}$	0	0	$P^{(0)}_{\alpha}$

. (55)

E_{α} and Σ_{α} can be used to define two new operators $P^{(\pm)}_{\alpha}$:

$$P^{(\pm)}_{\alpha} = \frac{E_{\alpha} \pm \Sigma_{\alpha}}{2} \ . \tag{56}$$

Note that the $P^{(\sigma)}_{\alpha}$ for $\sigma \in Z_{-1,1} = \{-1, 0, 1\}$ satisfy the following multiplication table:

$$\begin{array}{c|ccc} & P_\alpha^{(+)} & P_\alpha^{(-)} & P_\alpha^{(0)} \\ \hline P_\alpha^{(+)} & P_\alpha^{(+)} & 0 & 0 \\ P_\alpha^{(-)} & 0 & P_\alpha^{(-)} & 0 \\ P_\alpha^{(0)} & 0 & 0 & P_\alpha^{(0)} \end{array} \quad . \tag{57}$$

Thus, the $P_\alpha^{(\sigma)}$ are orthogonal projection operators.

It is easy to show using the definitions of $P_\alpha^{(\sigma)}$ and λ_σ for $\sigma \in Z_{-1,1}$ that

$$K_{\underline{ab}}^\alpha = \sum_{\sigma \in Z_{-1,1}} \lambda_\sigma P_\alpha^{(\sigma)} \ . \tag{58}$$

Thus,

$$\ln K_{\underline{ab}}^\alpha = \sum_{\sigma \in Z_{-1,1}} \ln(\lambda_\sigma) P_\alpha^{(\sigma)} \ . \tag{59}$$

Technically, we should also add a term $\ln(0) diag(0^{\#D_v+1}, 1^{\#3-D_v})$ to the right hand side of the last equation to account for the $3 - D_v$ zero eigenvalues of $K_{\underline{ab}}^\alpha$. However, we can safely ignore this infinite summand if we only use $\ln K_{\underline{ab}}^\alpha$ in expressions where it is multiplied times $\rho_{\underline{ab}}$. The infinite summand is annihilated when $\ln K_{\underline{ab}}^\alpha$ is multiplied times $\rho_{\underline{ab}}$.

5.2 $R_{\underline{ab}}^\alpha$ Calculations

To find $R_{\underline{ab}}^\alpha$, we need to calculate the partial traces of $K_{\underline{ab}}^\alpha$. One gets

$$K_{\underline{b}}^\alpha = \mathrm{tr}_{\underline{a}} K_{\underline{ab}}^\alpha = \frac{1}{N_{\underline{\alpha}}} \left(\frac{1}{2} + \vec{n}^\alpha \cdot \vec{B} \right) \ , \tag{60}$$

and

$$K_{\underline{a}}^\alpha = \mathrm{tr}_{\underline{b}} K_{\underline{ab}}^\alpha = \frac{1}{N_{\underline{\alpha}}} \left(\frac{1}{2} + (\mathcal{F}_2 \vec{n}^\alpha) \cdot \vec{A} \right) \ , \tag{61}$$

where

$$\vec{n}^\alpha = q \vec{v}^\alpha \ . \tag{62}$$

Therefore,

$$R_{\underline{ab}}^\alpha = \frac{K_{\underline{a}}^\alpha K_{\underline{b}}^\alpha}{w_\alpha} = \frac{1}{N_{\underline{\alpha}}} \left(\frac{1}{2} + \vec{n}^\alpha \cdot \vec{B} \right) \left(\frac{1}{2} + (\mathcal{F}_2 \vec{n}^\alpha) \cdot \vec{A} \right) \ . \tag{63}$$

We also need to know $\ln R_{\underline{ab}}^\alpha$. Using Appendix A, one finds

$$\begin{aligned} \ln R_{\underline{ab}}^\alpha &= -\ln N_{\underline{\alpha}} + \ln \left(\frac{1}{2} + \vec{n}^\alpha \cdot \vec{B} \right) + \ln \left(\frac{1}{2} + (\mathcal{F}_2 \vec{n}^\alpha) \cdot \vec{A} \right) \\ &= -\ln N_{\underline{\alpha}} + \sum_{\xi \in Bool} \ln \left(\frac{1}{2} + (-1)^\xi Y \right) \mathcal{P}_\xi \ , \end{aligned} \tag{64}$$

where

$$\mathcal{P}_\xi = \pi(|\xi_{\vec{n}^\alpha}\rangle_{\underline{b}}) + \pi(|\xi_{\mathcal{F}_2\vec{n}^\alpha}\rangle_{\underline{a}}) \tag{65}$$

for $\xi \in Bool$.

At this point we have calculated $\ln R^\alpha_{\underline{ab}}$, but we have not yet expressed it in the desired form, as a matrix in the Bell representation. To do this, we need to find the matrix elements in the Bell basis of the projectors $\pi(|\xi_{\vec{r}}\rangle_{\underline{b}})$ and $\pi(|\xi_{\vec{r}}\rangle_{\underline{b}})$ for $\xi \in Bool$. These matrix elements can be found using the techniques discussed in Section 3. One finds:

$$\begin{aligned} \pi(|\xi_{\vec{r}}\rangle_{\underline{b}}) &= \frac{1}{2}\left[1 + (-1)^\xi \vec{B}\cdot\hat{r}\right] \\ &= \frac{1}{2} + \frac{(-1)^\xi}{2}\begin{bmatrix} 0 & i\hat{r}^T \\ -i\hat{r} & i(\hat{r}\times\cdot) \end{bmatrix}, \end{aligned} \tag{66}$$

and

$$\begin{aligned} \pi(|\xi_{\vec{r}}\rangle_{\underline{a}}) &= \frac{1}{2}\left[1 + (-1)^\xi \vec{A}\cdot\hat{r}\right] \\ &= \frac{1}{2} + \frac{(-1)^\xi}{2}\begin{bmatrix} 0 & i(\mathcal{F}_2\hat{r})^T \\ -i\mathcal{F}_2\hat{r} & -i(\mathcal{F}_2\hat{r}\times\cdot) \end{bmatrix}. \end{aligned} \tag{67}$$

Putting all this together, we get

$$\ln R^\alpha_{\underline{ab}} = -\ln N_{\underline{\alpha}} + \ln\left((\frac{1}{2}+Y)(\frac{1}{2}-Y)\right) + \ln\left(\frac{\frac{1}{2}+Y}{\frac{1}{2}-Y}\right)\begin{bmatrix} 0 & i\hat{n}^{\alpha T} \\ -i\hat{n}^\alpha & 0 \end{bmatrix}. \tag{68}$$

5.3 E_{pure} and E_{mixed} Calculations

Recall from Ref.[1] that the following Lagrangian $\mathcal{L}$ must be minimized to obtain both E_{pure} and E_{mixed}:

$$\mathcal{L} = l_K - l_R\,, \tag{69}$$

where

$$l_K = \sum_\alpha \mathrm{tr}_{\underline{ab}}(K^\alpha_{\underline{ab}} \ln K^\alpha_{\underline{ab}})\,, \tag{70}$$

and

$$l_R = \sum_\alpha \mathrm{tr}_{\underline{ab}}(K^\alpha_{\underline{ab}} \ln R^\alpha_{\underline{ab}})\,. \tag{71}$$

Using the results of previous sections, one finds

$$l_K = -\ln N_{\underline{\alpha}} + \sum_{\sigma=\pm} \lambda'_\sigma \ln(\lambda'_\sigma) + (D_v - 1)\lambda'_0 \ln(\lambda'_0) \ , \tag{72}$$

where $\lambda'_\sigma = N_{\underline{\alpha}}\lambda_\sigma$ for $\sigma \in Z_{-1,1}$, and the λ_σ are just the eigenvalues of $K^\alpha_{\underline{ab}}$ that we found earlier. One also finds that

$$l_R = -\ln N_{\underline{\alpha}} - 2h_e(\frac{1}{2} + Y) \ . \tag{73}$$

Putting all this together, we get

$$\mathcal{L} = \left\{ \begin{array}{l} \sum_{\sigma=\pm}(\frac{u}{2} + \sigma X)\ln(\frac{u}{2} + \sigma X) \\ +(D_v - 1)\epsilon m_1 \ln(\epsilon m_1) \\ +2h_e(\frac{1}{2} + Y) \end{array} \right. \ . \tag{74}$$

Next we will use Eq.(74) to calculate entanglement $E = min(\mathcal{L})/(2\ln(2))$ for pure and mixed minimizations.

(case 1)Pure Min.

In the case of pure minimization, one has $q = \sqrt{m_0 m_1}$ and $\epsilon = 0$. Thus, the auxiliary parameters defined by Eqs.(47) reduce to: $\eta = D_v$, $u = 1$, $k = (m_0 - D_v m_1)/2$, $Y = \sqrt{m_0(D_v m_1)} = \sqrt{m_0(1 - m_0)}$, and $X = 1/2$. Thus, from Eq.(74), we get

$$E_{pure} = h\left(\frac{1}{2} + Y\right) \ . \tag{75}$$

If we define the *concurrence* C for this case to be:

$$C = |2m_0 - 1| \ , \tag{76}$$

then Eq.(75) can be rewritten as in Ref.[2]:

$$E_{pure} = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \ . \tag{77}$$

(case 2)Mixed Min.

For mixed minimization, the constraints $q = \sqrt{m_0 m_1}$ and $\epsilon = 0$ are no longer required in order to make $K^\alpha_{\underline{ab}}$ separable. We can choose q and ϵ so as to minimize $\mathcal{L}$ given by Eq.(74). Treating $\mathcal{L}$ as a function of q and ϵ and setting its partials to zero, we get the following two constraints

$$\frac{\partial \mathcal{L}}{\partial \epsilon} = 0 \ \Rightarrow \ \left\{ \begin{array}{ll} \text{either} & \ln(\epsilon m_1) = \sum_{\sigma=\pm}(\frac{1}{2} - \frac{\sigma k}{2X})\ln(\frac{u}{2} + \sigma X) \\ \text{or} & m_1(D_v - 1) = 0 \end{array} \right. \ , \tag{78a}$$

and

$$\frac{\partial \mathcal{L}}{\partial q} = 0 \ \Rightarrow \ \frac{1}{2X}\ln\left(\frac{\frac{u}{2} - X}{\frac{u}{2} + X}\right) = \frac{1}{Y}\ln\left(\frac{\frac{1}{2} - Y}{\frac{1}{2} + Y}\right) \ . \tag{78b}$$

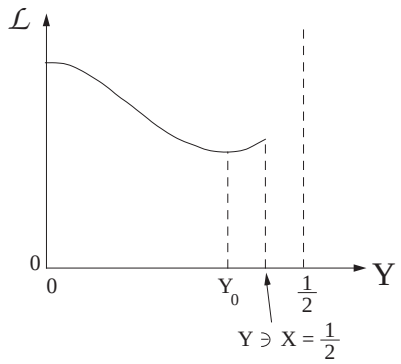

Figure 1: Plot of $\mathcal{L}$ versus Y at fixed k when $D_v = 1$. See Eq.(80).

In general,

$$E_{mixed} = \frac{\mathcal{L}}{2\ln 2} \ , \tag{79}$$

with $\mathcal{L}$ given by Eq.(74), subject to the constraints Eqs.(78).

Eqs.(78) with $D_v \neq 1$ constitute a system of two nonlinear equations which one would like to solve for the two unknowns ϵ and q. As far as we know, this "$\epsilon - q$ System" cannot be solved exactly in closed form– its roots can only be found using numerical techniques. See Appendix B if interested in an approximate analytical solution of the system.

The case of mixed min simplifies considerably when $D_v = 1$. In that case $K^{\alpha}_{\underline{ab}}$ and $\mathcal{L}$ are independent of ϵ, so we need only minimize over q. $\mathcal{L}$ reduces to

$$\mathcal{L} = -h_e\left(\frac{1}{2} + \sqrt{k^2 + Y^2}\right) + 2h_e\left(\frac{1}{2} + Y\right) \ . \tag{80}$$

Fig.1 is a plot of $\mathcal{L}$ versus Y at fixed k, according to Eq.(80). The largest possible Y value corresponds to pure minimization. $\mathcal{L}$ does not achieve its minimum at that endpoint, but rather at a smaller value of Y, which we call Y_0 in Fig.1. Y_0 can be determined by solving Eq.(78b) for Y as a function of k. We conclude that for $D_v = 1$,

$$E_{mixed} = -(\frac{1}{2})h(\frac{1}{2} + \sqrt{k^2 + Y_0^2}) + h(\frac{1}{2} + Y_0) \ . \tag{81}$$

5.4 $\Delta_{\underline{ab}}$ Calculations

Next, let us calculate $\Delta_{\underline{ab}}$ for the cases of mixed and pure minimizations.

(case 1)Pure Min.

The entanglement operator for pure min satisfies:

$$\Delta^{pure}_{\underline{ab}}|\psi_\alpha\rangle = (\ln K^{\alpha}_{\underline{ab}} - \ln R^{\alpha}_{\underline{ab}})|\psi_\alpha\rangle \ , \tag{82}$$

where

$$|\psi_\alpha\rangle = \begin{pmatrix} \sqrt{m_0} \\ -i\sqrt{m_1}\vec{v}^\alpha \end{pmatrix} . \tag{83}$$

Note that $\langle\psi_\alpha|\psi_\alpha\rangle = 1$. Using the results of previous sections, one can express the right hand side of Eq.(82) as follows. Define auxiliary quantities f, M and a by

$$f = \sqrt{\frac{1-m_0}{m_0}} , \tag{84}$$

$$M = -\ln\left(\frac{\frac{1}{2}+m_0 f}{\frac{1}{2}-m_0 f}\right) diag\left[f, \left(\frac{1}{f}\right)^{\#D_v}, 0^{\#3-D_v}\right] , \tag{85}$$

and

$$a = -\ln\left((\frac{1}{2}+m_0 f)(\frac{1}{2}-m_0 f)\right) . \tag{86}$$

Then

$$\ln R^\alpha_{\underline{ab}}|\psi_\alpha\rangle = (-\ln N_{\underline{\alpha}} - M - a)|\psi_\alpha\rangle . \tag{87}$$

Since

$$\ln K^\alpha_{\underline{ab}}|\psi_\alpha\rangle = -\ln(N_{\underline{\alpha}})|\psi_\alpha\rangle , \tag{88}$$

we get

$$\Delta^{pure}_{\underline{ab}} = M + a . \tag{89}$$

Using this value for $\Delta^{pure}_{\underline{ab}}$ and the value for E_{pure} that we obtained in Section 5.3, and also using the constraints $q = \sqrt{m_0 m_1}$, $\epsilon = 0$, one can check that

$$(2\ln 2)E_{pure} = \mathrm{tr}(\rho_{\underline{ab}}\Delta^{pure}_{\underline{ab}}) . \tag{90}$$

(case 2)Mixed Min.

The entanglement operator for mixed min satisfies:

$$\Delta^{mixed}_{\underline{ab}} = \ln K^\alpha_{\underline{ab}} - \ln R^\alpha_{\underline{ab}} . \tag{91}$$

Using the results of previous sections, one can express the right hand side of Eq.(91) as follows. Define the auxiliary quantities M and a by

$$M = diag\left[\sum_{\sigma=\pm}\left(\frac{1}{2}+\frac{\sigma k}{2X}\right)\ln\left(\frac{u}{2}+\sigma X\right), (\ln(\epsilon m_1))^{\#D_v}, 0^{\#3-D_v}\right] , \tag{92}$$

$$a = -\ln\left((\frac{1}{2}+Y)(\frac{1}{2}-Y)\right) \ . \tag{93}$$

Then

$$\Delta^{mixed}_{\underline{ab}} = M + a \ . \tag{94}$$

Using this value for $\Delta^{mixed}_{\underline{ab}}$, and the value for E_{mixed} that we obtained in Section 5.3, and also using the constraints Eqs.(78), one can check that

$$(2\ln 2)E_{mixed} = \mathrm{tr}(\rho_{\underline{ab}}\Delta^{mixed}_{\underline{ab}}) \ . \tag{95}$$

6 Implications

If Ω is an orthogonal matrix and we replace our vectors $\vec{v}^{\alpha}$ by $\Omega\vec{v}^{\alpha}$, then $K^{\alpha}_{\underline{ab}}$ changes but the value of the entanglement doesn't. Thus, there is a continuum of possible $K^{\alpha}_{\underline{ab}}$'s that minimizes $\mathcal{L}(K,K)$. It is convenient to define a *pure-min-entanglement orbit* (ditto, *mixed-min-entanglement orbit*) as a set of all $K^{\alpha}_{\underline{ab}} \in \mathcal{K}_{pure}$ (ditto, $K^{\alpha}_{\underline{ab}} \in \mathcal{K}_{mixed}$) which are stationary points of $\mathcal{L}(K,K)$, and which give the same value for $\mathcal{L}(K,K)$. Only one pure-min (ditto, mixed-min) orbit is a global minimum of $\mathcal{L}(K,K)$. Any $K^{\alpha}_{\underline{ab}}$ (or any orbit) whose $\Delta_{\underline{ab}}$ is independent of α, will be said to be α-*insensitive* (or just insensitive for short).

In previous sections, we found a pure-min insensitive $K^{\alpha}_{\underline{ab}}$ whose concurrence is $|2m_0-1|$, regardless of whether $m_0 > \frac{1}{2}$ or not. On the other hand, in Ref.[2], Bennett et al found a different $K^{\alpha}_{\underline{ab}}$ whose concurrence is zero when $m_0 \leq \frac{1}{2}$. (In fact, Ref.[2] shows that *any* Bell Mixture, not just the Werner states that we are considering here, must have zero entanglement of formation when the largest weight $m_0 \leq \frac{1}{2}$). Thus, for the Werner states that we are considering here, there exist at least two pure-min insensitive orbits when $m_0 < \frac{1}{2}$.

In Ref.[1], we claimed that there is only one orbit, the global minimum of $\mathcal{L}(K,K)$, that is insensitive. This paper has given a counterexample to that claim. It appears from this paper that the insensitivity condition is necessary for the global minimum orbit, but it is not always sufficient. This leads one to wonder why and when the sufficiency part of the proof in Ref.[1] breaks down. The breakdown may be due to the fact that the proof treats in a cavalier manner the infinities produced by taking the log of zero eigenvalues.

If we generalize our ansatz for $K^{\alpha}_{\underline{ab}}$ so that the $\vec{v}^{\alpha}$'s can be complex, can we find any more insensitive orbits? We try to answer this question in Appendices C and D. We find that going from real to complex $\vec{v}^{\alpha}$'s yields new insensitive orbits in the pure min but not in the mixed min cases. We also find that for the $\vec{v}^{\alpha}$-complex ansatz, there exist a countable number of pure-min insensitive orbits, and each of these corresponds to a stationary point of the pre-concurrence.

A Appendix: Eigensystem of 2 Dimensional Hermitian matrix

Consider any 2 by 2 Hermitian matrix $\tilde{n}$. One can always express it as $\tilde{n} = n_0 + \vec{n}\cdot\vec{\sigma}$, where n_0 and $\vec{n}$ are real. The eigensystem of $\tilde{n}$ follows immediately from the following easily proven identity:

$$n_0 + \vec{n}\cdot\vec{\sigma} = (n_0 + \sqrt{\vec{n}^2})\left(\frac{1+\hat{n}\cdot\vec{\sigma}}{2}\right) + (n_0 - \sqrt{\vec{n}^2})\left(\frac{1-\hat{n}\cdot\vec{\sigma}}{2}\right) , \tag{96}$$

where $\hat{n} = \vec{n}/\sqrt{\vec{n}^2}$. Define

$$P_\pm = \frac{1 \pm \hat{n}\cdot\vec{\sigma}}{2} . \tag{97}$$

Then

$$P_+P_- = P_-P_+ = 0 , \tag{98}$$

and

$$(P_\pm)^2 = P_\pm . \tag{99}$$

Thus, P_+ and P_- are the projectors onto the two eigenspaces of $\tilde{n}$ with respective eigenvalues $n_0 + \sqrt{\vec{n}^2}$ and $n_0 - \sqrt{\vec{n}^2}$.

An alternative, more tedious way of finding the eigensystem of $\tilde{n}$ is to rotate the equations $\sigma_z|0\rangle = |0\rangle$ and $\sigma_z|1\rangle = -|1\rangle$. Define a rotation vector $\vec{\theta}$ by:

$$\vec{\theta} = \theta\frac{\hat{z}\times\vec{n}}{|\hat{z}\times\vec{n}|} , \quad \theta = \arccos\frac{n_3}{|\vec{n}|} , \tag{100}$$

The spin up and down states along the $\vec{n}$ direction can be obtained in terms of those along the $\hat{z}$ by:

$$|0_{\vec{n}}\rangle = \exp(-i\frac{\vec{\sigma}\cdot\vec{\theta}}{2})|0\rangle , \tag{101}$$

and

$$|1_{\vec{n}}\rangle = \exp(-i\frac{\vec{\sigma}\cdot\vec{\theta}}{2})|1\rangle . \tag{102}$$

One can show that the projectors $P_\pm$ defined by Eq.(97) satisfy:

$$P_+ = |0_{\vec{n}}\rangle\langle 0_{\vec{n}}| , \tag{103}$$

and

$$P_- = |1_{\vec{n}}\rangle\langle 1_{\vec{n}}| . \tag{104}$$

B Appendix: Approximate Solution of $\epsilon - q$ System

In this appendix, we will assume $D_v > 1$. Eqs.(78) constitute a system of two nonlinear equations which one would like to solve for the two unknowns ϵ and q. As far as we know, this "$\epsilon - q$ System" cannot be solved exactly in closed form– its roots can only be found using numerical techniques. Nevertheless, as we will show next, it is possible to get approximate analytical expressions for its roots.

It is easy to show that the $\epsilon - q$ System is solved exactly by $(\epsilon, q) = (1, 0)$. This root, however, does not yield the global minimum mixed-min orbit, so it is of little interest to us. It can be rejected if we restrict our attention to roots for which the regulator q is nonzero.

From the numerical results obtained with Causa Común and discussed in Ref.[1], we expect that the $\epsilon - q$ System has a second root which is very close to the pure min case, for which $(\epsilon, q) = (0, \sqrt{m_0 m_1})$. The rest of this appendix will be devoted to finding an approximate value for this second root. Taylor expansion at $(\epsilon, q) = (0, \sqrt{m_0 m_1})$ is not possible since both equations of the $\epsilon - q$ System are non-analytic at that point. Another type of approximation is called for.

We begin by rewriting the $\epsilon - q$ System in the following equivalent form:

$$\epsilon m_1 = \left(\frac{u}{2} + X\right)^{\frac{1}{2} - \frac{k}{2X}} \left(\frac{u}{2} - X\right)^{\frac{1}{2} + \frac{k}{2X}} , \tag{105a}$$

$$\left(\frac{\frac{u}{2} - X}{\frac{u}{2} + X}\right)^{Y} = \left(\frac{\frac{1}{2} - Y}{\frac{1}{2} + Y}\right)^{2X} . \tag{105b}$$

When $\epsilon \approx 0$ and $q \approx \sqrt{m_0 m_1}$, one has that $u \approx u^0 \equiv 1$, $k \approx k^0 \equiv m_0 - \frac{1}{2}$, $Y \approx Y^0 \equiv \sqrt{m_0(1 - m_0)}$, and $X \approx X^0 \equiv \frac{1}{2}$. Our approximation consists of replacing Eqs.(105) by the following two equations:

$$\epsilon m_1 = \left(\frac{u^0}{2} + X^0\right)^{\frac{1}{2} - \frac{k^0}{2X^0}} \left(\frac{u}{2} - X\right)^{\frac{1}{2} + \frac{k^0}{2X^0}} , \tag{106a}$$

$$\left(\frac{\frac{u}{2} - X}{\frac{u^0}{2} + X^0}\right)^{Y^0} = \left(\frac{\frac{1}{2} - Y}{\frac{1}{2} + Y^0}\right)^{2X^0} . \tag{106b}$$

In effect, what we are doing is setting $\epsilon = 0$ and $q = \sqrt{m_0 m_1}$ everywhere in Eqs.(105) except where a very small number (namely, $\frac{u}{2} - X$ and $\frac{1}{2} - Y$) appears raised to a power. Since $\frac{1}{2} - Y^0 = 0$ iff $m_0 = \frac{1}{2}$, this approximation is expected to work best in the vicinity of $m_0 = \frac{1}{2}$. Define the quantity ρ by:

$$\rho = m_0 m_1 \eta - q^2 D_v = (\frac{u}{2})^2 - X^2 . \tag{107}$$

Eqs.(106) can be expressed in terms of ρ as follows:

$$\epsilon m_1 = \rho^{m_0} \ , \tag{108a}$$

$$\rho^{Y^0}(\frac{1}{2} + Y^0) = \frac{1}{2} - q\sqrt{D_v} \ . \tag{108b}$$

Eqs.(108) imply

$$m_0 m_1 \eta = m_0 m_1 \left[D_v - \epsilon(D_v - 1)\right] = m_0 m_1 \left[D_v - \frac{\rho^{m_0}}{m_1}(D_v - 1)\right] \ , \tag{109a}$$

and

$$q^2 D_v = \left[\frac{1}{2} - \rho^{Y^0}(\frac{1}{2} + Y^0)\right]^2 \ . \tag{109b}$$

Now the left hand sides of Eqs.(109) are just the two terms whose difference defines ρ. Therefore,

$$\rho = (Y^0)^2 \left[1 - \frac{\rho^{m_0}}{m_1}(1 - \frac{1}{D_v})\right] - \left[\frac{1}{2} - \rho^{Y^0}(\frac{1}{2} + Y^0)\right]^2 \ . \tag{110}$$

Motivated by the last equation, we define a function f of ρ by:

$$f(\rho) = -\rho + (Y^0)^2 \left[1 - \frac{\rho^{m_0}}{1 - m_0}(D_v - 1)\right] - \left[\frac{1}{2} - \rho^{Y^0}(\frac{1}{2} + Y^0)\right]^2 \ . \tag{111}$$

Eqs.(108) can rewritten as follows, so that they express ϵ and q in terms of ρ:

$$\epsilon = \frac{\rho^{m_0}}{1 - m_0} D_v \ , \tag{112a}$$

$$q = \frac{1}{\sqrt{D_v}} \left[\frac{1}{2} - \rho^{Y^0}(\frac{1}{2} + Y^0)\right] \ . \tag{112b}$$

Clearly, given any ρ_{root} for which $f(\rho_{root}) = 0$, one can use Eqs.(112) to calculate a point $(\epsilon, q)_{root}$ that approximately satisfies the original $\epsilon - q$ System. If $(\epsilon, q)_{root}$ yields values for u, k, X and Y that are close to u^0, k^0, X^0 and Y^0, respectively, then our original assumptions are vindicated and we say the approximation is self consistent.

Let $m_0 = \frac{1}{2} + \delta m$. We wrote a simple Excel spreadsheet in which we plotted $f(\rho)$ as a function of ρ, with inputs m_0 and D_v. Fig.2 shows schematically what we found. For $D_v = 2$, $f(\rho)$ has a zero regardless of the sign of δm. For $D_v = 3$, $f(\rho)$ has a zero for $\delta m > 0$ but not for $\delta m < 0$.

We also calculated analytical approximations for f, and for its first and second derivatives, in the limit of small ρ and small $|\delta m|$:

$$f(0) = (Y^0)^2 - \frac{1}{4} \leq 0 \ , \tag{113}$$

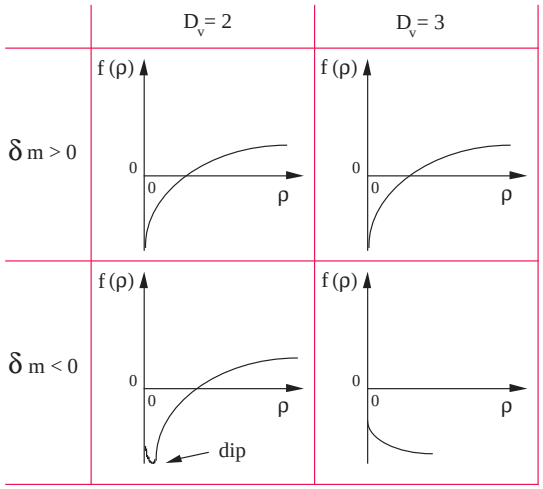

Figure 2: Schematic plots of $f(\rho)$. Four cases depicted depending on whether D_v equals 2 or 3 and whether δm is greater or smaller than zero.

$$\frac{\partial f}{\partial \rho} \to \frac{1}{\sqrt{\rho}}\left[\frac{-1}{4}(D_v-1)\rho^{\delta m}+\frac{1}{2}\right] \to \begin{cases} \frac{1}{2\sqrt{\rho}} & \text{if } \delta m > 0 \\ \frac{-(D_v-1)}{4\rho^{\frac{1}{2}+|\delta m|}} & \text{if } \delta m < 0 \end{cases} , \tag{114}$$

$$\frac{\partial^2 f}{\partial \rho^2} \to \frac{1}{\rho^{\frac{3}{2}}}\left[\frac{-1}{4}(D_v-1)(\delta m-\frac{1}{2})\rho^{\delta m}-\frac{1}{4}\right] \to \begin{cases} \frac{-1}{4\rho^{\frac{3}{2}}} & \text{if } \delta m > 0 \\ \frac{(D_v-1)}{8\rho^{\frac{3}{2}+|\delta m|}} & \text{if } \delta m < 0 \end{cases} . \tag{115}$$

C Appendix: Stationary Points of the Pre-concurrence

Define the *pre-concurrence amplitude* γ_θ by

$$\gamma_\theta = \sum_{j=0}^{n} e^{i\theta_j} m_j \ , \tag{116}$$

where $m_j \geq 0$ for all $j \in Z_{0,n}$, $\sum_{j=0}^{n} m_j = 1$, the θ_j are real numbers, and we fix $\theta_0 = 0$. Next define the *pre-concurrence* C_θ by

$$C_\theta = |\gamma_\theta| \ . \tag{117}$$

The global minimum of C_θ (at $n = 3$) over all phases $\vec{\theta}$ is called the *concurrence*. This global minimum arose in our calculation of the pure min entanglement of a two qubit

system. One wonders if C_θ has other stationary points, and whether they play a role in our theory. In this appendix, we will find the stationary points of C_θ. In Appendix D, we will show that such stationary points are indeed very relevant to our theory.

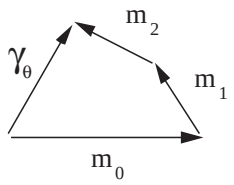

Figure 3: γ_θ is a "fractured unit vector".

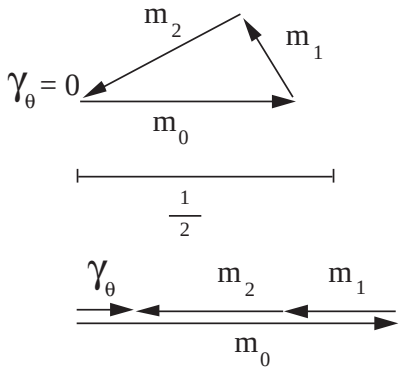

Figure 4: $\min|\gamma_\theta|$ equals zero if $m_0 \leq \frac{1}{2}$ and $m_0 - \sum_{j=1}^n m_j$ if $m_0 > \frac{1}{2}$.

From Fig.3, we see that γ is a "fractured unit vector" in the complex plane: it equals the vector sum of segments whose lengths m_j add to one. From this geometrical picture, the smallest possible length for γ_θ is clear. As shown in Fig.4, if $m_0 \leq \frac{1}{2}$, then $\min|\gamma_\theta| = 0$. If $m_0 > \frac{1}{2}$, then $\min|\gamma_\theta| = m_0 - \sum_{j=1}^n m_j$. Are there any other stationary points of C_θ? Yes.

Any stationary point $\vec{\theta}$ of C_θ must satisfy:

$$\delta C_\theta^2 = \delta(\gamma_\theta \gamma_\theta^*) = i \sum_{j=1}^n \delta\theta_j m_j (e^{i\theta_j}\gamma_\theta^* - e^{-i\theta_j}\gamma_\theta) = 0 \; . \tag{118}$$

Hence,

$$m_j |\gamma_\theta| \sin(\theta_j - \angle(\gamma_\theta)) = 0 \; , \tag{119}$$

for all $j \in Z_{1,n}$. The last equation is satisfied iff either $\gamma_\theta = 0$, or, for all $j \in Z_{1,n}$ such that $m_j \neq 0$, $\theta_j = \angle(\gamma_\theta) + \pi b_j + 2\pi n_j$, where $b_j \in Bool$ and the n_j are integers. Hence, the set of stationary values which C_θ can assume is given by

$$\left\{ |\sum_{j=0}^{n}(-1)^{b_j} m_j| : b_j \in Bool, b_0 = 0 \right\} , \tag{120}$$

together with $C_\theta = 0$, which, however, is only possible when $m_0 \leq \frac{1}{2}$.

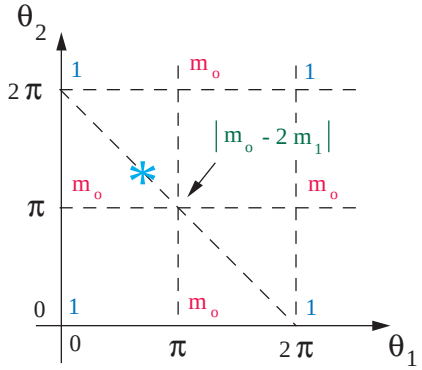

Figure 5: Values of $C_\theta(\theta_1, \theta_2)$ shown in color. C_θ is periodic in θ_1 and θ_2. Its minimum value is $|m_0 - 2m_1|$ if $m_0 \geq \frac{1}{2}$ and zero otherwise. $C_\theta = 0$ (possible only if $m_0 \leq \frac{1}{2}$) is indicated by an asterisk.

Fig.5 shows a plot of

$$C_\theta(\theta_1, \theta_2) = |m_0 + m_1(e^{i\theta_1} + e^{i\theta_2})| \; ; \tag{121}$$

that is, C_θ when $n = 2$ and $m_1 = m_2$.

D Appendix: Complex $\vec{v}^\alpha$'s

Recall that an orbit is a set of all $K^\alpha_{\underline{ab}}$ which give a common stationary value for $\mathcal{L}(K, K)$. And if $\Delta_{\underline{ab}}$ for an orbit is independent of α, we say that the orbit is insensitive. As was pointed out in Section 6, the results of this paper imply that there can be more than one pure-min insensitive orbit. If we generalize our ansatz for $K^\alpha_{\underline{ab}}$ so that the $\vec{v}^\alpha$'s can be complex, can we find any more insensitive orbits? We know that the global minimum of the pre-concurrence corresponds to an insensitive orbit. Do the other stationary points of the pre-concurrence also correspond to insensitive orbits? In this appendix we will show that going from real to complex $\vec{v}^\alpha$'s yields new insensitive orbits in the pure min but not in the mixed min cases. We will also show that for the $\vec{v}^\alpha$-complex ansatz, there exist a countable number of pure-min insensitive orbits, and each of these corresponds to a stationary point of the pre-concurrence.

Let U^α be defined by

$$U^\alpha = diag(e^{i\phi^\alpha_1}, e^{i\phi^\alpha_2}, e^{i\phi^\alpha_3}) \; , \tag{122}$$

where the ϕ_j^α are real. Let

$$\bar{U}^\alpha = \begin{bmatrix} 1 & 0 \\ 0 & U^\alpha \end{bmatrix} . \tag{123}$$

One can define complex vectors $\vec{\zeta}^\alpha$ in terms the real ones $\vec{v}^\alpha$ by:

$$\vec{\zeta}^\alpha = U^\alpha \vec{v}^\alpha . \tag{124}$$

Note that by virtue of Eqs.(43), $\vec{\zeta}^{\alpha\dagger}\vec{\zeta}^\alpha = D_v$, $\sum_\alpha \vec{\zeta}^\alpha = 0$, and $\sum_\alpha \vec{\zeta}^\alpha \vec{\zeta}^{\alpha\dagger} = N_{\underline{\alpha}} I_v$. Our new ansatz for $K^\alpha_{\underline{ab}}$ is defined in terms of the old one by:

$$\tilde{K}^\alpha_{\underline{ab}} = \bar{U}^\alpha K^\alpha_{\underline{ab}} (\bar{U}^\alpha)^* = (K^\alpha_{\underline{ab}})_{\vec{v}^\alpha \to \vec{\zeta}^\alpha} . \tag{125}$$

Henceforth, as we have just done, when we need to distinguish between a new quantity and the corresponding old one, we will indicate the new one by a tilde. (By old we mean with the $\vec{v}^\alpha$ real, and by new, with the $\vec{v}^\alpha$ complex.) Sometimes, if it is clear from the context that we are speaking of new quantities, we will drop the tildes. Alternatively, sometimes we will introduce a new symbol for a new quantity to distinguish it from the corresponding old one, as we did by introducing the symbol $\vec{\zeta}^\alpha$ to represent the new $\vec{v}^\alpha$.

It is convenient to define z_0 and $\vec{z}^\alpha$ by:

$$z_0 = \sqrt{m_0} , \tag{126}$$

$$\vec{z}^\alpha = -i\sqrt{m_1}\vec{\zeta}^\alpha . \tag{127}$$

Note that $(z_0)^2 + \vec{z}^\alpha \cdot \vec{z}^{\alpha *} = 1$.

One can define a *concurrence amplitude* γ by

$$\begin{aligned} \gamma &= (z_0)^2 + (\vec{z}^\alpha)^2 \\ &= m_0 - m_1 (\vec{\zeta}^\alpha)^2 \\ &= m_0 - m_1 \sum_{j=1}^{3} e^{i2\phi_j^\alpha} (v_j^\alpha)^2 . \end{aligned} \tag{128}$$

The *concurrence* C is defined as the absolute value of γ:

$$C = |\gamma| . \tag{129}$$

Note that in general, γ and C depend on α, However, as will soon become apparent, they must be independent of α for any insensitive orbit.

Next we will calculate the new $\tilde{K}^\alpha_{\underline{ab}}$, $\tilde{R}^\alpha_{\underline{ab}}$ and $\tilde{\Delta}_{\underline{ab}}$. The calculation is very similar to that of the old $K^\alpha_{\underline{ab}}$, $R^\alpha_{\underline{ab}}$ and $\Delta_{\underline{ab}}$ presented in previous sections.

(a) $\tilde{K}^\alpha_{\underline{ab}}$ Calculations

Since $\tilde{K}^{\alpha}_{\underline{ab}}$ and $K^{\alpha}_{\underline{ab}}$ are simply related by a unitary transformation, it is clear from previous results for $K^{\alpha}_{\underline{ab}}$ (see Section 5.1) that

$$\ln \tilde{K}^{\alpha}_{\underline{ab}} = -\ln N_{\underline{\alpha}} + \sum_{\sigma \in Z_{-1,1}} \ln(\lambda'_{\sigma}) \tilde{P}^{(\sigma)}_{\alpha} \ , \tag{130}$$

where

$$\tilde{P}^{(\sigma)}_{\alpha} = \bar{U}^{\alpha} P^{(\sigma)}_{\alpha} (\bar{U}^{\alpha})^{*} = (P^{(\sigma)}_{\alpha})_{\vec{v}^{\alpha} \to \vec{\zeta}^{\alpha}} \ . \tag{131}$$

Written more explicitly, this becomes

$$\ln \tilde{K}^{\alpha}_{\underline{ab}} = \left\{ \begin{array}{l} -\ln N_{\underline{\alpha}} \\ +\sum_{\sigma=\pm} \ln(\frac{u}{2} + \sigma X) \left(\begin{array}{cc} \frac{1}{2} + \frac{\sigma k}{2X} & \frac{\sigma i q \vec{\zeta}^{\alpha\dagger}}{2X} \\ \frac{-\sigma i q \vec{\zeta}^{\alpha}}{2X} & (\frac{1}{2} - \frac{\sigma k}{2X}) \frac{\vec{\zeta}^{\alpha} \vec{\zeta}^{\alpha\dagger}}{D_v} \end{array} \right) \\ +\ln(\epsilon m_1) \left(\begin{array}{cc} 0 & 0 \\ 0 & I_v - \frac{\vec{\zeta}^{\alpha} \vec{\zeta}^{\alpha\dagger}}{D_v} \end{array} \right) \end{array} \right. \ . \tag{132}$$

(b) $\tilde{R}^{\alpha}_{\underline{ab}}$ Calculations

We proceed as we did in Section 5.2 where we calculated $R^{\alpha}_{\underline{ab}}$. Now we find that:

$$\tilde{K}^{\alpha}_{\underline{b}} = \mathrm{tr}_{\underline{a}} \tilde{K}^{\alpha}_{\underline{ab}} = \frac{1}{N_{\underline{\alpha}}} \left(\frac{1}{2} + (\vec{n}^{\alpha} + \Delta \vec{n}^{\alpha}) \cdot \vec{B} \right) \ , \tag{133}$$

and

$$\tilde{K}^{\alpha}_{\underline{a}} = \mathrm{tr}_{\underline{b}} \tilde{K}^{\alpha}_{\underline{ab}} = \frac{1}{N_{\underline{\alpha}}} \left(\frac{1}{2} + [\mathcal{F}_2(\vec{n}^{\alpha} - \Delta \vec{n}^{\alpha})] \cdot \vec{A} \right) \ , \tag{134}$$

where

$$\vec{n}^{\alpha} = \frac{i}{2} (\vec{z}^{\alpha} - \vec{z}^{\alpha *}) z_0 \frac{q}{\sqrt{m_0 m_1}} \ , \tag{135}$$

and

$$\Delta \vec{n}^{\alpha} = \frac{i}{2} (\vec{z}^{\alpha} \times \vec{z}^{\alpha *})(1 - \epsilon) \ . \tag{136}$$

Note that the $\Delta \vec{n}^{\alpha}$ enters with opposite signs in Eqs.(133) and (134). Proceeding as we did in Section 5.2, we finally find

$$\ln \tilde{R}^{\alpha}_{\underline{ab}} = \left\{ \begin{array}{l} -\ln N_{\underline{\alpha}} \\ +\ln \left((\frac{1}{2} + \tilde{Y})(\frac{1}{2} - \tilde{Y}) \right) \\ +\ln \left(\frac{\frac{1}{2} + \tilde{Y}}{\frac{1}{2} - \tilde{Y}} \right) \frac{1}{\tilde{Y}} \left[\begin{array}{cc} 0 & i \vec{n}^{\alpha T} \\ -i \vec{n}^{\alpha} & i \Delta \vec{n}^{\alpha} \times \cdot \end{array} \right] \end{array} \right. \ , \tag{137}$$

where

$$\tilde{Y} = |\vec{n}^{\alpha} + \Delta\vec{n}^{\alpha}| \ . \tag{138}$$

Note that in general, $\tilde{Y}$ depends on α, However, as will soon become apparent, $\tilde{Y}$ must be independent of α for any insensitive orbit. Note also that $\vec{n}^{\alpha} \cdot \Delta\vec{n}^{\alpha} = 0$ so $\tilde{Y}^2 = |\vec{n}^{\alpha}|^2 + |\Delta\vec{n}^{\alpha}|^2$. After some algebra, one can show that:

$$|\vec{n}^{\alpha}|^2 = \left(\frac{q^2}{m_0 m_1}\right) \frac{m_0}{2}(1-\gamma_r) \ , \tag{139}$$

and

$$\begin{aligned} |\Delta\vec{n}^{\alpha}|^2 &= (1-\epsilon)^2 \left\{ \frac{1-|\gamma|^2}{4} - \frac{m_0}{2}(1-\gamma_r) \right\} \\ &= \frac{(1-\epsilon)^2}{4} \left\{ (1-\gamma_r)[\gamma_r - (2m_0-1)] - \gamma_i^2 \right\} \ , \end{aligned} \tag{140}$$

where γ_r and γ_i are the real and imaginary parts of γ. We see that when $\gamma = 1$ for all α, $\vec{n}^{\alpha} = \Delta\vec{n}^{\alpha} = 0$ for all α. In the other extreme, when $\gamma = 0$, one gets $|\vec{n}^{\alpha}|^2 = \frac{q^2}{2m_1}$ and $|\Delta\vec{n}^{\alpha}|^2 = \frac{(1-\epsilon)^2}{4}(1-2m_0)$. If besides $\gamma = 0$, one assumes the pure min constraints $q = \sqrt{m_0 m_1}$, $\epsilon = 0$, then $|\vec{n}^{\alpha}|^2 = m_0/2$, $|\Delta\vec{n}^{\alpha}|^2 = (1-2m_0)/4$, so $\tilde{Y} = |\vec{n}^{\alpha} + \Delta\vec{n}^{\alpha}| = \frac{1}{2}$. Recall that for the real-$\vec{v}^{\alpha}$ ansatz, one has $\gamma = |2m_0 - 1|$ and $Y = \sqrt{m_0(1-m_0)}$, and therefore, $\gamma = 0$ implies $m_0 = Y = \frac{1}{2}$. We see that in the complex-$\vec{v}^{\alpha}$ ansatz, it is possible to have $\gamma = 0$, $\tilde{Y} = \frac{1}{2}$, and $m_0 \neq \frac{1}{2}$.

(c) $\tilde{\Delta}_{\underline{ab}}$ Calculations

For the mixed min case, $\tilde{\Delta}^{mixed}_{\underline{ab}}$ is simply

$$\tilde{\Delta}^{mixed}_{\underline{ab}} = \ln \tilde{K}^{\alpha}_{\underline{ab}} - \ln \tilde{R}^{\alpha}_{\underline{ab}} \ , \tag{141}$$

where $\ln \tilde{K}^{\alpha}_{\underline{ab}}$ and $\ln \tilde{R}^{\alpha}_{\underline{ab}}$ have just been calculated.

For the pure min case, $(\epsilon, q) = (0, \sqrt{m_0 m_1})$ and $|\psi_{\alpha}\rangle$ is defined by Eq.(83), which can be expressed in terms of z_0 and $\vec{z}^{\alpha}$ as

$$|\psi_{\alpha}\rangle = \begin{pmatrix} z_0 \\ \vec{z}^{\alpha} \end{pmatrix} \ . \tag{142}$$

One finds

$$\begin{aligned} \tilde{\Delta}^{pure}_{\underline{ab}} |\psi_{\alpha}\rangle &= (\ln \tilde{K}^{\alpha}_{\underline{ab}} - \ln \tilde{R}^{\alpha}_{\underline{ab}}) |\psi_{\alpha}\rangle \\ &= \left\{ \begin{array}{l} -\ln\left((\frac{1}{2} + \tilde{Y})(\frac{1}{2} - \tilde{Y}) \right) \\ -\ln\left(\frac{\frac{1}{2}+\tilde{Y}}{\frac{1}{2}-\tilde{Y}} \right)^{\frac{1}{2\tilde{Y}}} \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1+\gamma(U^{\alpha})^{2*} \end{pmatrix} \end{array} \right\} |\psi_{\alpha}\rangle \ . \end{aligned} \tag{143}$$

As pointed out earlier in this appendix, if $q = \sqrt{m_0 m_1}$, $\epsilon = 0$, and $\gamma = 0$, then $\tilde{Y} = \frac{1}{2}$. In this limit, Eq.(143) gives $\tilde{\Delta}^{pure} = 0$, as expected.

A simple Lemma: Consider an insensitive orbit for which $\vec{n}^\alpha$ and $\Delta\vec{n}^\alpha$ are independent of α. If this premise is satisfied, then $q = 0$ and $\epsilon = 1$. The premise is true in particular when $\gamma = 1$ for all α. Proof: By Eq.(137) and the premise, $\ln \tilde{R}^\alpha_{\underline{ab}}$ is independent of α. Since $\ln \tilde{R}^\alpha_{\underline{ab}}$ and $\Delta_{\underline{ab}}$ are independent of α, $\ln \tilde{K}^\alpha_{\underline{ab}}$ (and therefore $\tilde{K}^\alpha_{\underline{ab}}$) must be independent of α too. From the ansatz form for $\tilde{K}^\alpha_{\underline{ab}}$, it is clear that if $\tilde{K}^\alpha_{\underline{ab}}$ is independent of α, then $q = 0$ and $\epsilon = 1$. If $\gamma = 1$ for all α, then by Eqs.(139) and (140), $\vec{n}^\alpha = \Delta\vec{n}^\alpha = 0$. Therefore, $\vec{n}^\alpha$ and $\Delta\vec{n}^\alpha$ are independent of α. QED

We end this section by finding all insensitive orbits that are possible according to the just derived formulas for $\ln \tilde{K}^\alpha_{\underline{ab}}$, $\ln \tilde{R}^\alpha_{\underline{ab}}$ and $\tilde{\Delta}_{\underline{ab}}$.

Pure Min Case:

Our arguments are based on Eq.(143), which we restate here for convenience:

$$\begin{aligned} \tilde{\Delta}^{pure}_{\underline{ab}}|\psi_\alpha\rangle &= (\ln \tilde{K}^\alpha_{\underline{ab}} - \ln \tilde{R}^\alpha_{\underline{ab}})|\psi_\alpha\rangle \\ &= \left\{ \begin{array}{l} -\ln\left((\frac{1}{2}+\tilde{Y})(\frac{1}{2}-\tilde{Y})\right) \\ -\ln\left(\frac{\frac{1}{2}+\tilde{Y}}{\frac{1}{2}-\tilde{Y}}\right)\frac{1}{2\tilde{Y}}\begin{pmatrix} 1-\gamma & 0 \\ 0 & 1+\gamma(U^\alpha)^{2*} \end{pmatrix} \end{array} \right\}|\psi_\alpha\rangle \ . \end{aligned} \tag{144}$$

We will henceforth call this equation Eq.A. Note the following.

For the right hand side of Eq.A to be independent of α (for an open set of m_0 values), $\tilde{Y}$ and γ must be independent of α.

If $\gamma = 0$, then we must have $m_0 \leq \frac{1}{2}$, as explained in Appendix C. According to that Appendix, $\gamma = 0$ is one of the possible stationary values of the pre-concurrence.

If $\gamma \neq 0$, then from Eq.A, $(U^\alpha)^2$ must be real and independent of α. Thus, $e^{i2\phi^\alpha_j} = (-1)^{b_j}$ for $j \in Z_{1,3}$, where $b_j \in Bool$ is independent of α. This gives $\gamma = m_0 - \sum_{j=1}^3 (-1)^{b_j} m_j$, where $m_j = m_1 (v^\alpha_j)^2$. In the examples given by Eqs.(44), $(v^\alpha_j)^2$ is independent of α for all j.

If $b_j = 1$ for all j, then $\gamma = 1$. By the Lemma proven previously, this implies that $q = 0$ and $\epsilon = 1$. We can exclude this case because $q = 0$ and $\epsilon = 1$ would not give (except for some special values of m_0) a $K^\alpha_{\underline{ab}}$ of the form $w_\alpha|\psi^\alpha\rangle\langle\psi^\alpha|$.

Comparing the above results with Appendix C, we see that there are a countable number of pure-min insensitive orbits, and each of these corresponds to a stationary point of the pre-concurrence.

Mixed Min Case:

Our arguments are based on Eqs.(132), Eqs.(137) and Eqs.(141), which we restate here for convenience:

$$\ln \tilde{K}^{\alpha}_{\underline{ab}} = \left\{ \begin{array}{l} -\ln N_{\underline{\alpha}} \\ +\sum_{\sigma=\pm} \ln(\frac{u}{2}+\sigma X) \left(\begin{array}{cc} \frac{1}{2}+\frac{\sigma k}{2X} & \frac{\sigma i q \vec{\zeta}^{\alpha\dagger}}{2X} \\ \frac{-\sigma i q \vec{\zeta}^{\alpha}}{2X} & (\frac{1}{2}-\frac{\sigma k}{2X})\frac{\vec{\zeta}^{\alpha}\vec{\zeta}^{\alpha\dagger}}{D_v} \end{array} \right) \\ +\ln(\epsilon m_1) \left(\begin{array}{cc} 0 & 0 \\ 0 & I_v - \frac{\vec{\zeta}^{\alpha}\vec{\zeta}^{\alpha\dagger}}{D_v} \end{array} \right) \end{array} \right. . \tag{145a}$$

$$\ln \tilde{R}^{\alpha}_{\underline{ab}} = \left\{ \begin{array}{l} -\ln N_{\underline{\alpha}} \\ +\ln\left((\frac{1}{2}+\tilde{Y})(\frac{1}{2}-\tilde{Y})\right) \\ +\ln\left(\frac{\frac{1}{2}+\tilde{Y}}{\frac{1}{2}-\tilde{Y}}\right)\frac{1}{\tilde{Y}} \left[\begin{array}{cc} 0 & i\vec{n}^{\alpha T} \\ -i\vec{n}^{\alpha} & i\Delta\vec{n}^{\alpha}\times\cdot \end{array} \right] \end{array} \right. , \tag{145b}$$

$$\tilde{\Delta}^{mixed}_{\underline{ab}} = \ln \tilde{K}^{\alpha}_{\underline{ab}} - \ln \tilde{R}^{\alpha}_{\underline{ab}} . \tag{145c}$$

Henceforth, we will call these 3 equations A, B and C, respectively.

Subtracting entries (1,0), (2,0) and (3,0) of the right hand sides of Eqs.A and B, we get:

$$\ln\left(\frac{\frac{u}{2}-X}{\frac{u}{2}+X}\right)\frac{iq\vec{\zeta}^{\alpha}}{2X} - \ln\left(\frac{\frac{1}{2}+\tilde{Y}}{\frac{1}{2}-\tilde{Y}}\right)\frac{1}{\tilde{Y}}(-i\vec{n}^{\alpha}) . \tag{146}$$

This vector must be independent of α. Since $\vec{n}^{\alpha} = \frac{q}{2}(\vec{\zeta}^{\alpha} + \vec{\zeta}^{\alpha *})$, we must have $\lambda_1\vec{\zeta}^{\alpha} + \lambda_2\vec{\zeta}^{\alpha *} = \vec{v}$, where λ_1, λ_2 are real numbers, and λ_1, λ_2, $\vec{v}$ are independent of α. But then $\sum_\alpha(\lambda_1\vec{\zeta}^{\alpha} + \lambda_2\vec{\zeta}^{\alpha *}) = 0 = \vec{v}$, so $\lambda_1\vec{\zeta}^{\alpha} + \lambda_2\vec{\zeta}^{\alpha *} = 0$. In other words, $(\lambda_1 + \lambda_2 e^{-i2\phi^{\alpha}_j})v^{\alpha}_j = 0$ for all α and j. Assume λ_1 and λ_2 are non-zero for an open set of m_0 values. Also assume that $v^{\alpha}_j \neq 0$ for all α and $j \in Z_{1,D_v}$ (this is true for the examples given in Eqs.(44). Then $e^{-i2\phi^{\alpha}_j}$ must be real and independent of α and $j \in Z_{1,D_v}$. If $e^{-i2\phi^{\alpha}_j} = -1$, then $\gamma = 1$. But by the Lemma proven previously, this implies that $q = 0$ and $\epsilon = 1$. We can exclude this case if we require the regulator q to be non-zero. Thus, there appears to be just one mixed-min insensitive orbit, the one found in Section 5 for the real-$\vec{v}^{\alpha}$ ansatz.

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